The Tower of States and the Entanglement Spectrum in a Coplanar Antiferromagnet

Louk Rademaker, Max Metlitski
KITP, UCSB

I. What is the 'tower of states'?

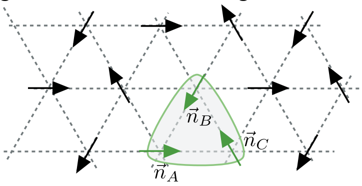
A system that exhibits spontaneous continuous symmetry breaking (SSB) in the thermodynamic limit will, for any finite system size, have a **unique ground state**. How then can any **finite system show signatures** of its symmetry breaking fate? Anderson pointed out that such a finite size system contains a **'tower of states'** or 'thin spectrum': **eigenstates with an energy** $\mathcal{O}(1/N)$ that vanishes in the thermodynamic limit, and who have a degeneracy structure that reveals the order parameter symmetry.
This tower of states can be used to **discover the symmetry broken state numerically**.
In general, the **ground state entanglement** can reveal the low-energy spectrum of a phase.
For SSB-systems, the **entanglement spectrum** will have the 'tower of states' structure. This is shown for collinear magnets in [1], in this work we show it for coplanar antiferromagnets.

II. Coplanar Antiferromagnet

In a coplanar antiferromagnet the order parameter is not a vector but rather an **SO(3) rotation**.
The prime example is the Heisenberg antiferromagnet

$$H = J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$$

on a triangular lattice with the magnetic order



defined by the three vectors n_A, n_B and n_C.

III. SO(3) Nonlinear Sigma Model

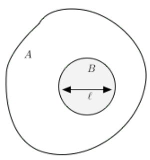
The coplanar antiferromagnet can be described by a **nonlinear sigma model**, where the vector **a** lives on the 3-sphere,

$$S = \frac{1}{2} \int d^2x \int_0^\beta d\tau \left(\chi_\parallel (\partial_\tau \vec{a})^2 + \rho_\parallel (\vec{\nabla} \vec{a})^2 + (\chi_\perp - \chi_\parallel)(\vec{a}\gamma^1 \partial_\tau \vec{a})^2 + (\rho_\perp - \rho_\parallel)(\vec{a}\gamma^1 \vec{\nabla} \vec{a})^2 \right)$$

where the vector field $\gamma^1 \vec{a}$ defines the **anisotropy** between in-plane and out-of-plane spin waves.

IV. The Entanglement properties computed in Four Steps

IV.A. To compute the **ground state wavefunction**, we split off the k=0 components of the a-field into the constant component $\vec{n}_0$ and the spin wave fluctuations $\pi_i(\vec{x})$. The wavefunction for the $\vec{n}_0$ component is a singlet, for the spin waves we get an anisotropic Gaussian wavepacket,
$\psi[\vec{n}_0, \pi_i] \propto \exp\left(-\frac{1}{4} \int d^2x d^2y \pi_i(\vec{x}) c_{ij} Q(\vec{x}, \vec{y}) \pi_j(\vec{y}) \right)$ with inverse propagator $Q(\vec{x}, \vec{y}) = \sum_{\vec{k} \neq 0} 2|\vec{k}| e^{i\vec{k}\cdot(\vec{x}-\vec{y})}$ and c_{ij} the spin wave velocity matrix.



IV.B. The **Reduced Density Matrix** can be obtained by integrating out the degrees of freedom on a subregion *B*, starting from the full density matrix $\rho(\vec{a}, \vec{a}') = \psi[\vec{a}]\psi^*[\vec{a}']$. Therefore we expand the fields around the north pole of the 3-sphere, $a^0(\vec{x}) = \sqrt{1 - \pi_i(\vec{x})^2/\rho_i}, a^i(\vec{x}) = \pi_i(\vec{x})/\sqrt{\rho_i}$. After Gaussian integration over subregion *B* we obtain

$$\rho_A[\pi_A, \pi'_A] \propto \exp\left[-\frac{1}{8}(\pi_A - \pi'_A)\hat{c} Q_{AA} (\pi_A - \pi'_A) - \frac{1}{8}(\pi_A + \pi'_A)\hat{c} \left(Q_{AA} - Q_{AB} Q_{BB}^{-1} Q_{BA} \right) (\pi_A + \pi'_A) \right]$$

IV.C. The k=0 component of the reduced density matrix now displays the **tower of states**. To see that, we can write out the zero momentum part in terms of an **entanglement Hamiltonian,**

$$H^E_{tos} = \frac{1}{2I\rho_\parallel c_\parallel} \sum_{\alpha > \beta} L^2_{\alpha\beta} + \frac{(\rho_\perp c_\perp - \rho_\parallel c_\parallel)}{8I\rho_\parallel c_\parallel (2\rho_\parallel c_\parallel - \rho_\perp c_\perp)} (\mathrm{Tr}\, L\gamma^1)^2$$

where *I* depends on the size of subregion *B*,

$$I = \frac{1}{4} \int_A d^2x d^2y Q(\vec{x}, \vec{y}) \sim \ell \log(\ell/a)$$

The lowest eigenstates are now of the form $|\{j\} m_1 m_2\rangle$ with energies $E_{j,m_1,m_2} \sim I^{-1} \left(a j(j+1) + b (m_2)^2 \right)$, which is lower than the spin wave eigenstates with energy $\mathcal{O}(\log \ell)^{-1}$.

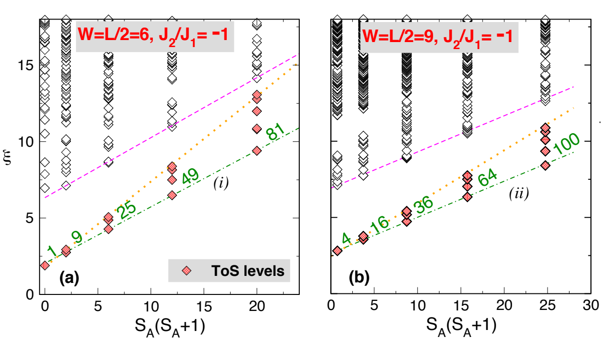
IV.D. The **entanglement entropy** can now be computed.
The spin waves are just free bosons, and hence give an **area law** contribution to the entanglement entropy.
The k=0 component of the reduced density matrix can be raised to the *n*-th power

$$\rho^n_A[\pi_0, \pi''_0] \sim n^{-3/2} \left(\frac{2\pi}{I} \right)^{3(n-1)/2} (\det \hat{c}\hat{\rho})^{-(n-1)/2} \exp\left[-\frac{I}{2n} (\pi_0 - \pi''_0)^i c_{ij} (\pi_0 - \pi''_0)^j \right]$$

so that the **Von Neumann entanglement entropy** obtains an extra **logarithmic term** due to the tower of states,

$$S^E_{tos} \sim \frac{3}{2} \log \ell + \mathrm{const.}$$

where the universal prefactor *N/2* counts the **number of Goldstone modes** of the symmetry broken state.



V. Comparison to numerics

The structure of the states in the entanglement spectrum that we found using the nonlinear sigma model is the same as found by Kolley et al [2] using DMRG.

VI. Conclusion

Using an *SO(3)* nonlinear sigma model, we show that the ground state entanglement spectrum of a coplanar antiferromagnet displays the 'tower of states' structure. Consistent with earlier results for $O(N)$ models, the entanglement entropy counts the number of Goldstone modes of the broken symmetry state.

References:

[1] Metlitski, Grover, arXiv:1112.5166 (2011).
[2] Kolley et al, PRB 88, 144426 (2013).
[3] Rademaker, Metlitski, to be published.